

Derivation of Markovian Master Equation by Renormalization Group Method

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We present a derivation of the Markovian master equation by the renormalization group method. Starting from a naive perturbative solution of the von Neumann equation, the reduced density matrix with the coarse grained time steps is obtained using the assumption of short correlation time of the bath field. Then by applying the renormalization group method, we show that the dependence of the specific initial time on the perturbative solution can be removed and the Markovian semigroup master equation in the Gorini–Kossakowski–Lindblad–Sudarshan (GKLS) form is obtained in the weak coupling limit.

1. Introduction

The quantum dynamics of an open system cannot in general be represented in terms of unitary time evolution. One useful method of investigating the open quantum system is to formulate its dynamics by means of an appropriate equation of motion for its density matrix; this is a quantum master equation. As one of the typical examples of the open quantum system, the particle detector model^{1,2} is widely accepted as a tool for exploring the quantum nature of the considering system. It is composed of detector variables (system) with internal states that interact with external quantum fields (bath). This model has been applied to phenomena of particle creations in curved spacetimes (Unruh effect and Hawking radiation). This model is also used to detect the entanglement of quantum fields by investigating the correlation between two independent detectors interacting with the quantum fields.^{3,4,5,6,7,8}

The state of the detector is obtained by tracing out the field degrees of freedoms which interacts with the detector (reduced density matrix). As a result, the evolution of the system

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is determined by the dynamical map \mathcal{E} , which is not unitary in general.

$$\begin{array}{ccc}
 \rho_T(0) = \rho(0) \otimes \rho_B & \xrightarrow{\text{unitary evolution}} & \rho_T(t) = U(t, 0)[\rho(0) \otimes \rho_B]U^\dagger(t, 0) \\
 \text{tr}_B \downarrow & & \text{tr}_B \downarrow \\
 \rho(0) & \xrightarrow{\text{dynamical map}} & \rho(t) = \mathcal{E}(t)\rho(0)
 \end{array}$$

To describe physically allowable processes, the dynamical map should preserve the complete positivity and normalization of probabilities in the course of evolution.

The dynamical map can be represented by the master equation that determines the evolution of the system. In general, the master equation becomes an integro-differential equation, and the state at the specific time depends on the past evolution (non-Markovian nature). However, it is possible to recover the Markovian property of the master equation by assuming suitable time scales: the decay time of the bath correlation function t_B is sufficiently shorter than the relaxation time scale of the system t_S . By combining this assumption of the time scales with the so-called secular approximation (rotation wave approximation), which neglects transitions via system energy nonconserving processes, it can be shown that the resulting master equation has the Gorini–Kossakowski–Lindblad–Sudarshan (GKLS) form and preserves the trace and complete positivity of the state in the course of evolution.^{9,10} It is known that the Markovian master equation, which preserves the trace and complete positivity, should have the GKLS form and generates the dynamical semigroup.¹¹

As the derivation of a master equation that does not rely on the secular approximation, the time coarse graining method was proposed.^{12,13,5,14} By introducing a coarse graining time scale, which is assumed to be longer than the decay time scale of the bath correlation function, it is possible to derive the GKLS master equation without assuming the secular approximation. The purpose of this paper is to present an alternative derivation of the GKLS master equation using the renormalization group method.^{15,16}

The renormalization group method is applied as a tool of asymptotic analysis of differential equations.^{15,16} The naive perturbative solution of differential equations often yields secular terms due to resonance phenomena. The secular terms prevent us from obtaining approximate but global solutions. The renormalization group method is one of the techniques to circumvent the problem. Starting from a naive perturbative expansion, the secular divergence is absorbed into constants of integration contained in the zeroth-order solution by the renormalization procedure. The renormalized constants obey the renormalization group equation. We shortly review this method using the following example. Let us consider the van der Pol

equation

$$\frac{d^2x}{dt^2} + x = \epsilon(1 - x^2)\frac{dx}{dt}, \quad (1)$$

where ϵ is a constant. For a small ϵ , the naive perturbative solution up to the order ϵ is

$$x(t) = Ae^{it} + \epsilon \frac{t}{2} A(1 - |A|^2)e^{it} + \text{c.c} + \dots, \quad (2)$$

where A is a constant and nonresonant terms such as those proportional to $e^{\pm 3it}$ are not explicitly written. The perturbation fails beyond a time scale $\sim 1/\epsilon$ owing to the secular term, which grows as $\propto t$. To improve this naive perturbative solution, we introduce an arbitrary renormalization point τ by splitting t to $t - \tau + \tau$ and the renormalized constant $A(\tau)$ by $A = A(\tau) + \delta A(\tau)$, $\delta A(0) = 0$, where the counter term δA is chosen to absorb τ . Thus,

$$\delta A(\tau) = A(0) - A(\tau) \equiv -\epsilon \frac{\tau}{2} A(\tau)(1 - |A(\tau)|^2), \quad (3)$$

and the naive perturbative solution can be written as

$$x(t) = A(\tau)e^{it} + \epsilon \frac{t - \tau}{2} A(\tau)(1 - |A(\tau)|^2)e^{it} + \text{c.c} + \dots. \quad (4)$$

As the renormalization point τ is arbitrary, by equating τ and t , we obtain the renormalized solution $x(t) = A(t)e^{it}$ with $A(t)$ being the solution of the following renormalization group equation:

$$\frac{dA}{dt} = \frac{\epsilon}{2} A(1 - |A|^2). \quad (5)$$

This is an amplitude modulation equation that represents the slow dynamics of the system. The renormalized solution gives an approximated but global solution to Eq. (1) up to $O(\epsilon)$.

In this paper, starting from a naive perturbative solution of the von Neumann equation, we rederive the master equation in the GKLS form with the time coarse graining by the renormalization group method. We will show that the time coarse graining by the renormalization procedure naturally defines the semigroup structure for the renormalized constant in discretized time steps of the evolution of the reduced density matrix. We use the unit in which $\hbar = 1$ throughout the paper.

2. Derivation of Markovian master equation

2.1 Naive solution of the von Neumann equation

Concerning the derivation of a naive perturbative solution of the von Neumann equation, we basically follow the presentation by Benatti *et al.*⁵ The total system is composed of detector variables (system) interacting with quantum scalar fields (bath). The total Hamiltonian

is

$$H = H_0^S + H_0^B + \lambda V, \quad (6)$$

where H_0^S is the system Hamiltonian, H_0^B is the bath Hamiltonian and V is the interaction Hamiltonian with $V = \sum_A \sigma_A \Phi_A$, where σ_A is the system operator and Φ_A is the bath field. The strength of the interaction between the system and the bath is determined by the coupling constant λ , which is assumed to be small (weak coupling). The total density operator $\rho_T(t)$ in the Schrödinger picture obeys the von Neumann equation

$$\frac{d}{dt}\rho_T = -i[H_0 + \lambda V, \rho_T], \quad H_0 = H_0^S + H_0^B. \quad (7)$$

We aim to obtain an equation for the reduced density operator for the detector system

$$\rho(t) \equiv \text{Tr}_B\{\rho_T(t)\}. \quad (8)$$

First, we introduce the interaction picture of the density operator defined by

$$\tilde{\rho}_T(t) = U_0^\dagger(t, t_0)\rho_T(t)U_0(t, t_0), \quad U_0(t, t_0) = e^{-iH_0(t-t_0)},$$

where t_0 is an arbitrary initial time. Then, $\tilde{\rho}_T$ obeys

$$\frac{d}{dt}\tilde{\rho}_T = -i[\lambda \tilde{V}(t), \tilde{\rho}_T], \quad \tilde{V}(t) = U_0^\dagger(t, t_0)VU_0(t, t_0), \quad (9)$$

where $\tilde{V}(t)$ is the interaction representation of V . We integrate this equation from t_0 to t . After two iterations, the formal solution is given by

$$\begin{aligned} \tilde{\rho}_T(t) = & \tilde{\rho}_T(t_0) - i\lambda \int_{t_0}^t dt_1 [\tilde{V}(t_1), \tilde{\rho}_T(t_0)] \\ & - \lambda^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 [\tilde{V}(t_1), [\tilde{V}(t_2), \tilde{\rho}_T(t_0)]] + O(\lambda^3). \end{aligned} \quad (10)$$

We assume that during the evolution, the state of the total system is factorized

$$\tilde{\rho}_T(t) \approx \tilde{\rho}(t) \otimes \rho_B, \quad (11)$$

where $\rho_B = \rho_B(t_0)$. This assumption is justified because the interaction between the system and the bath is weak and the correlation between the system and the bath can be neglected when the bath time scale t_B is shorter than the system time scale t_S . As the bath contains an infinite number of degrees of freedom, the back action of the system on the bath is negligible and the evolved total state can be expressed as the product state (11). By taking the trace of the perturbative solution (10) with respect to the bath degrees of freedom, the reduced density operator for the system is

$$\tilde{\rho}(t) = \rho(t_0) - \lambda^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \text{Tr}_B \{ [\tilde{V}(t_1), [\tilde{V}(t_2), \rho(t_0) \otimes \rho_B(t_0)]] \}, \quad (12)$$

where we have assumed

$$\text{Tr}_B\{\tilde{V}(t)\rho(t_0) \otimes \rho_B(t_0)\} = 0. \quad (13)$$

If we do not consider the back action of the system on the bath and the bath variable evolves freely, the bath operators in the interaction picture are the same as those in the Heisenberg picture $\tilde{\Phi}_A(t) = \Phi_A(t)$. Thus, the condition (13) is equivalent to $\langle \Phi_A \rangle = 0$ because

$$\text{Tr}_B\{\tilde{V}(t)\rho(t_0) \otimes \rho_B(t_0)\} = \sum_A \tilde{\sigma}_A(t) \langle \Phi_A(t) \rangle = 0,$$

where the expectation value of the bath field $\Phi(t)$ (Heisenberg operator) is evaluated as

$$\langle \Phi(t) \rangle \equiv \text{Tr}_B\{\Phi(t)\rho_B(t_0)\}.$$

We rewrite the integral in Eq. (12). Let us define

$$A_{12} \equiv \text{Tr}_B\left\{\left[\tilde{V}(t_1), \left[\tilde{V}(t_2), \rho(t_0) \otimes \rho_B(t_0)\right]\right]\right\}, \quad (14)$$

and decompose the integral as

$$\int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 A_{12} = \frac{1}{4} \int_{t_0}^t dt_1 \int_{t_0}^t dt_2 (A_{12} + A_{21}) + \frac{1}{2} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 (A_{12} - A_{21}). \quad (15)$$

From now on, we omit the symbol \otimes for simplicity. The first integral is

$$\int_{t_0}^t dt_1 \int_{t_0}^t dt_2 \text{Tr}_B\left(\frac{1}{2}\{\tilde{V}(t_1)\tilde{V}(t_2), \rho(t_0)\rho_B(t_0)\} - \tilde{V}(t_1)\rho(t_0)\rho_B(t_0)\tilde{V}(t_2)\right), \quad (16)$$

and the second integral is

$$\begin{aligned} & \frac{1}{2} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \text{Tr}_B\left[[\tilde{V}(t_1), \tilde{V}(t_2)], \rho(t_0)\rho_B(t_0)\right] \\ &= \frac{1}{2} \int_{t_0}^t dt_1 \int_{t_0}^t dt_2 \text{sgn}(t_1 - t_2) \left[\text{Tr}_B(\rho_B(t_0)\tilde{V}(t_1)\tilde{V}(t_2)), \rho(t_0)\right]. \end{aligned} \quad (17)$$

Using $V = \sum_A \sigma_A \Phi_A$ and introducing the correlation function of the bath field by¹⁷

$$G_{A_1 A_2}(t_1 - t_2) = \text{Tr}_B\{\Phi_{A_1}(t_1)\Phi_{A_2}(t_2)\rho_B(t_0)\} = \langle \Phi_{A_1}(t_1)\Phi_{A_2}(t_2) \rangle, \quad (18)$$

we can rewrite the integral in the right-hand side of Eq. (12) as

$$\tilde{\rho}(t) - \tilde{\rho}(t_0) = \Delta t \times (-i[H_{12}, \rho_0] + \mathcal{L}[\rho_0]), \quad \Delta t = t - t_0, \quad \rho_0 = \rho(t_0), \quad (19)$$

where we have defined

$$\begin{aligned} H_{12} &= -i \frac{\lambda^2}{2\Delta t} \sum_{A_1, A_2} \int_0^{\Delta t} ds_1 \int_0^{\Delta t} ds_2 \text{sgn}(s_1 - s_2) G_{A_1 A_2}(s_1 - s_2) \sigma_{A_1}(s_1 + t_0) \sigma_{A_2}(s_2 + t_0), \\ \mathcal{L}[\rho_0] &= \frac{\lambda^2}{\Delta t} \sum_{A_1, A_2} \int_0^{\Delta t} ds_1 \int_0^{\Delta t} ds_2 G_{A_1 A_2}(s_1 - s_2) \left(\sigma_{A_2}(s_2 + t_0) \rho_0 \sigma_{A_1}(s_1 + t_0) \right. \end{aligned} \quad (20)$$

$$- \frac{1}{2} \{ \sigma_{A_1}(s_1 + t_0) \sigma_{A_2}(s_2 + t_0), \rho_0 \} \}. \quad (21)$$

In Eqs. (20) and (21), the system variable $\tilde{\sigma}_A(t)$ in the interaction picture is replaced by $\sigma_A(t)$ in the Heisenberg picture whose evolutions are determined by the free system Hamiltonian H_0^S . This is justified within perturbation up to $O(\lambda^2)$. The time dependence of the system variable is determined by the free Hamiltonian H_0^S as

$$\sigma_A(t) = e^{iH_0^S(t-t_0)} \sigma_A e^{-iH_0^S(t-t_0)} = \sum_B u_{AB}(t-t_0) \sigma_B, \quad u_{AB}(0) = 1, \quad (22)$$

where the specific form of the function $u_{AB}(t)$ depends on the system Hamiltonian. Using this relation, we obtain

$$H_{12} = \sum_{B_1, B_2} H_{B_1 B_2} \sigma_{B_1} \sigma_{B_2}, \quad (23)$$

$$\mathcal{L}[\rho_0] = \sum_{B_1, B_2} C_{B_1 B_2} \left(\sigma_{B_2} \rho_0 \sigma_{B_1} - \frac{1}{2} \{ \sigma_{B_1} \sigma_{B_2}, \rho_0 \} \right), \quad (24)$$

with

$$H_{B_1 B_2} = -i \frac{\lambda^2}{2\Delta t} \sum_{A_1, A_2} \int_0^{\Delta t} ds_1 \int_0^{\Delta t} ds_2 \operatorname{sgn}(s_1 - s_2) G_{A_1 A_2}(s_1 - s_2) u_{A_1 B_1}(s_1) u_{A_2 B_2}(s_2), \quad (25)$$

$$C_{B_1 B_2} = \frac{\lambda^2}{\Delta t} \sum_{A_1, A_2} \int_0^{\Delta t} ds_1 \int_0^{\Delta t} ds_2 G_{A_1 A_2}(s_1 - s_2) u_{A_1 B_1}(s_1) u_{A_2 B_2}(s_2). \quad (26)$$

As Eq. (19) determines $\tilde{\rho}(t)$ from the specific initial state at t_0 , we cannot obtain the master equation even though if we take the limit $\Delta t \rightarrow 0$. To convert (19) to the Markovian master equation, we introduce the time coarse graining and assume the short correlation time of the bath. Then we apply the renormalization group method^{15,16} to recover the Markovian master equation.

2.2 Time coarse graining and renormalization group

The naive perturbative solution (19) has the following structure:

$$\begin{aligned} & \tilde{\rho}(t) - \tilde{\rho}(t_0) \\ &= \lambda^2 \Delta t \times \frac{1}{\Delta t} \sum_{A_1, A_2, B_1, B_2} \int_0^{\Delta t} ds_1 \int_0^{\Delta t} ds_2 G_{A_1 A_2}(s_1 - s_2) u_{A_1 B_1}(s_1) u_{A_2 B_2}(s_2) F_{B_1 B_2}[s_1 - s_2, \rho_0] \\ &\equiv \lambda^2 \Delta t \times \mathcal{D}[\rho_0], \end{aligned} \quad (27)$$

where $F_{B_1 B_2}$ represents a combination of the following terms:

$$\operatorname{sgn}(s_1 - s_2) \sigma_{B_1} \sigma_{B_2} \rho_0, \operatorname{sgn}(s_1 - s_2) \rho_0 \sigma_{B_1} \sigma_{B_2}, \sigma_{B_1} \sigma_{B_2} \rho_0, \rho_0 \sigma_{B_1} \sigma_{B_2}, \sigma_{B_2} \rho_0 \sigma_{B_1}.$$

Let us introduce a coarse graining time scale Δ and consider the evolution of the state by this time step. Using (27), the evolution of n time steps from t_0 to $t_0 + n\Delta$ is

$$\begin{aligned} & \tilde{\rho}(t_0 + n\Delta) - \tilde{\rho}(t_0) \\ &= \lambda^2 \Delta \times \frac{1}{\Delta} \int_0^{n\Delta} ds_1 \int_0^{n\Delta} ds_2 \sum_{A_1, A_2, B_1, B_2} G_{A_1 A_2}(s_1 - s_2) u_{A_1 B_1}(s_1) u_{A_2 B_2}(s_2) F_{B_1 B_2}[s_1 - s_2, \rho_0] \\ &\equiv \lambda^2 \Delta \times \mathcal{D}_n[\rho_0]. \end{aligned} \quad (28)$$

At this point, we assume that the coarse graining step size Δ is sufficiently larger than the correlation time t_B of the bath field and the relation $t_B \ll \Delta$ holds. By introducing new integration variables as $\tau_1 = s_1 - s_2, \tau_2 = s_2 - s_1$, the integral of (28) is

$$\begin{aligned} & \int_0^{n\Delta} d\tau_1 \int_{\tau_1}^{n\Delta} ds_2 \sum G(\tau_1) u(\tau_1 + s_2) u(s_2) F[\tau_1] \\ & \quad + \int_0^{n\Delta} d\tau_2 \int_{\tau_2}^{n\Delta} ds_1 \sum G(-\tau_2) u(s_1) u(\tau_2 + s_1) F[-\tau_2] \\ & \approx \int_0^\infty d\tau \int_0^{n\Delta} ds \sum (G(\tau) u(s) u(s) F[\tau] + G(-\tau) u(s) u(s) F[-\tau]), \end{aligned} \quad (29)$$

where we have extended the upper bound of the τ_1, τ_2 integrals to infinity because the correlation function decays rapidly for $\tau_1, \tau_2 \neq 0$ and the main contribution to the integrals comes from the region $\tau_1, \tau_2 \approx 0$. Then, \mathcal{D}_n is reduced to the following form:

$$\mathcal{D}_n[\rho_0] = \frac{1}{\Delta} \int_0^{n\Delta} ds \sum_{A_1, A_2, B_1, B_2} u_{A_1 B_1}(s) u_{A_2 B_2}(s) \int_{-\infty}^{+\infty} d\tau G_{A_1 A_2}(\tau) F_{B_1 B_2}[\tau, \rho_0], \quad (30)$$

and it is possible to derive the following relations for $n \geq 1$:

$$\begin{aligned} (\mathcal{D}_n - \mathcal{D}_{n-1})[\rho_0] &= \frac{1}{\Delta} \int_{(n-1)\Delta}^{n\Delta} ds \sum u(s) u(s) \int_{-\infty}^{+\infty} d\tau G(\tau) F[\tau, \rho_0] \\ &= \frac{1}{\Delta} \int_0^\Delta ds \sum u(s + (n-1)\Delta) u(s + (n-1)\Delta) \int_{-\infty}^{+\infty} d\tau G(\tau) F[\tau, \rho_0] \\ &= e^{iH_0^S(n-1)\Delta} \mathcal{D}_1 \left[e^{-iH_0^S(n-1)\Delta} \rho_0 e^{iH_0^S(n-1)\Delta} \right] e^{-iH_0^S(n-1)\Delta}, \end{aligned} \quad (31)$$

where we have defined $\mathcal{D}_0 = 0$. Let us denote the state at $t_0 + n\Delta$ as $\tilde{\rho}_n = \tilde{\rho}(t_0 + n\Delta)$.

The perturbative solution (28) is represented as

$$\tilde{\rho}_n = C + (\lambda^2 \Delta) \mathcal{D}_n[C], \quad C = \tilde{\rho}_0 = \tilde{\rho}(t_0). \quad (32)$$

As the state in the interaction picture does not evolve in $O(\lambda^0)$, C is a constant of motion in this order. Hence, we regard the $O(\lambda^2)$ term in (32) as the secular term due to the interaction and apply the renormalization group method to remove the secular behaviour of the solution.

We introduce an arbitrary renormalization point k ($0 \leq k \leq n$). The renormalized constant C_k and the counter term δC_k are introduced as $C = C_k + \delta C_k$, $C_0 = C$, $\delta C_0 = 0$. The counter term is chosen so as to absorb the secular term of the solution:

$$\delta C_k = C_0 - C_k \equiv -(\lambda^2 \Delta) \mathcal{D}_k[C_0]. \quad (33)$$

By using the renormalized constant, the naive perturbative solution (32) can be written as

$$\tilde{\rho}_n = C_k + (\lambda^2 \Delta)(\mathcal{D}_n - \mathcal{D}_k)[C_0], \quad (0 \leq k \leq n). \quad (34)$$

Equation (33) defines a map $\mathcal{R}_k : C_0 \mapsto C_k$ ($0 \leq k$). As the renormalization point is arbitrary, the following two equations hold for two different renormalization points k_1, k_2 ($0 \leq k_1, k_2 \leq n$):

$$C_{k_1} - C_0 = (\lambda^2 \Delta)(\mathcal{D}_{k_1} - \mathcal{D}_0)[C_0], \quad C_{k_2} - C_0 = (\lambda^2 \Delta)(\mathcal{D}_{k_2} - \mathcal{D}_0)[C_0].$$

By subtracting these equations side by side, we obtain the following relation up to $O(\lambda^2)$:

$$C_{k_2} - C_{k_1} = (\lambda^2 \Delta)(\mathcal{D}_{k_2} - \mathcal{D}_{k_1})[C_{k_1}], \quad (0 \leq k_1 \leq k_2). \quad (35)$$

Thus, this equation defines a map $\mathcal{R}_{k_2-k_1} : C_{k_1} \mapsto C_{k_2}$ ($0 \leq k_2 - k_1$). As the map defined by (35) satisfies the composition law $\mathcal{R}_{k_2-k_1} \circ \mathcal{R}_{k_1} = \mathcal{R}_{k_2}$ ($k_1 \leq k_2$), it generates a discretized version of the dynamical semigroup. Therefore, the evolution of the system can be Markovian in the discretized (coarse grained) time steps. In Eq. (34), as the renormalization point k is arbitrary, we can choose $k = n$:

$$\tilde{\rho}_n = C_n. \quad (36)$$

This is the renormalized solution and the renormalized constant C_n satisfies the discretized version of the renormalization group equation (35). Combining (35) with (36) and using the relation (31), we obtain the following difference equation for $\tilde{\rho}_n$:

$$\begin{aligned} \tilde{\rho}_n - \tilde{\rho}_{n-1} &= (\lambda^2 \Delta)(\mathcal{D}_n - \mathcal{D}_{n-1})[\tilde{\rho}_{n-1}] \\ &= (\lambda^2 \Delta) e^{iH_0^S(n-1)\Delta} \mathcal{D}_1 \left[e^{-iH_0^S(n-1)\Delta} \tilde{\rho}_{n-1} e^{iH_0^S(n-1)\Delta} \right] e^{-iH_0^S(n-1)\Delta}. \end{aligned} \quad (37)$$

Now, we introduce a new time variable $\tau = \lambda^2 t$ and take the weak coupling limit $\lambda \rightarrow 0$. For $\tilde{\rho}(t)$, we define the time coarse grained state as

$$\tilde{\rho}^{\text{CG}}(\tau_n) = \tilde{\rho}(\tau_n/\lambda^2)|_{\lambda \rightarrow 0} = \tilde{\rho}_n|_{\lambda \rightarrow 0}. \quad (38)$$

To determine the evolution for the finite interval of time $\tau_n - \tau_0 = \lambda^2 n \Delta$, we must take $n \rightarrow \infty$.

In this limit,

$$\lim_{\lambda \rightarrow 0} \frac{\tilde{\rho}_n - \tilde{\rho}_{n-1}}{\lambda^2 \Delta} = \lim_{\lambda \rightarrow 0} \frac{\tilde{\rho}^{\text{CG}}(\tau_n) - \tilde{\rho}^{\text{CG}}(\tau_n - \lambda^2 \Delta)}{\lambda^2 \Delta} = \frac{d}{d\tau} \tilde{\rho}^{\text{CG}}(\tau), \quad (39)$$

and

$$\lim_{\lambda \rightarrow 0} \tilde{\rho}_{n-1} = \lim_{\lambda \rightarrow 0} \tilde{\rho}^{\text{CG}}(\tau_n - \lambda^2 \Delta) = \tilde{\rho}^{\text{CG}}(\tau_n), \quad (n-1)\Delta = \lim_{\lambda \rightarrow 0} \frac{1}{\lambda^2} (\tau_n - \lambda^2 \Delta - \tau_0) = (\tau_n - \tau_0)/\lambda^2.$$

Therefore, we obtain the differential equation for the state $\tilde{\rho}^{\text{CG}}(\tau)$:

$$\frac{d}{d\tau} \tilde{\rho}^{\text{CG}}(\tau) = e^{iH_0^S(\tau-\tau_0)/\lambda^2} \mathcal{D}_1 \left[e^{-iH_0^S(\tau-\tau_0)/\lambda^2} \tilde{\rho}^{\text{CG}}(\tau) e^{iH_0^S(\tau-\tau_0)/\lambda^2} \right] e^{-iH_0^S(\tau-\tau_0)/\lambda^2}. \quad (40)$$

The difference equation (37) is converted to the differential equation in the rescaled time variable τ with the small coupling limit. Hence, (40) represents the evolution of the time coarse grained state.

Returning back to the Schrödinger representation using $\tilde{\rho}^{\text{CG}}(t) = e^{iH_0^S(t-t_0)} \rho^{\text{CG}}(t) e^{-iH_0^S(t-t_0)}$, we obtain the following linear Markovian master equation:

$$\frac{d}{dt} \rho^{\text{CG}}(t) = -i[H_0^S, \rho^{\text{CG}}(t)] + \mathcal{D}_1[\rho^{\text{CG}}(t)], \quad (41)$$

where the bath-dependent contribution $\mathcal{D}_1[\rho^{\text{CG}}(t)]$ contains both a Hamiltonian and a dissipative term

$$\mathcal{D}_1[\rho^{\text{CG}}(t)] = -i[H_{12}^\Delta, \rho^{\text{CG}}(t)] + \mathcal{L}^\Delta[\rho^{\text{CG}}(t)], \quad (42)$$

with

$$H_{12}^\Delta = -\frac{i}{2} \sum_{B_1, B_2} H_{B_1 B_2}^\Delta \sigma_{B_1} \sigma_{B_2}, \quad (43)$$

$$\mathcal{L}^\Delta[\rho] = \sum_{B_1, B_2} C_{B_1 B_2}^\Delta \left(\sigma_{B_2} \rho \sigma_{B_1} - \frac{1}{2} \{ \sigma_{B_1} \sigma_{B_2}, \rho \} \right). \quad (44)$$

Thus, the master equation is

$$\frac{d}{dt} \rho^{\text{CG}}(t) = -i[H_0^S + H_{12}^\Delta, \rho^{\text{CG}}(t)] + \mathcal{L}^\Delta[\rho^{\text{CG}}(t)]. \quad (45)$$

The time-independent coefficients $H_{B_1 B_2}^\Delta, C_{B_1 B_2}^\Delta$ are given by (25) and (26) replacing $\Delta t \rightarrow \Delta$. These coefficients do not contain the initial time and the initial state and depend only on the coarse graining time Δ . Therefore, if the coefficients $C_{B_1 B_2}^\Delta$ form a positive matrix, the master equation (41) has the GKLS form and the state evolves preserving the trace and complete positivity. To show the positivity of the matrix $C_{B_1 B_2}^\Delta$ defined by (26), we must check the inequality $\sum_{B_1 B_2} \chi_{B_1}^* C_{B_1 B_2}^\Delta \chi_{B_2} \geq 0$ for an arbitrary vector χ_B . The form of the matrix is

$$C_{B_1 B_2}^\Delta = \frac{\lambda^2}{\Delta} \left\langle \sum_{A_1} \int_0^\Delta ds_1 \Phi_{A_1}(s_1) u_{A_1 B_1}(s_1) \sum_{A_2} \int_0^\Delta ds_2 \Phi_{A_2}(s_2) u_{A_2 B_2}(s_2) \right\rangle$$

$$\equiv \frac{\lambda^2}{\Delta} \langle L_{B_1} L_{B_2} \rangle, \quad (46)$$

where we introduced the operator

$$L_B \equiv \sum_A \int_0^\Delta \Phi_A(s) u_{AB}(s). \quad (47)$$

Without loss of generality, we can assume that the operators Φ_A and σ_A are Hermitian. Then the function $u_{AB}(t - t_0)$ representing the evolution of the system variable σ_A is real and the operator L_A is the Hermitian $L_B = L_B^\dagger$. Hence,

$$\begin{aligned} \sum_{B_1 B_2} \chi_{B_1}^* C_{B_1 B_2}^\Delta \chi_{B_2} &= \frac{\lambda^2}{\Delta} \sum_{B_1, B_2} \langle \chi_{B_1}^* L_{B_1} L_{B_2} \chi_{B_2} \rangle \\ &= \frac{\lambda^2}{\Delta} \left\langle \left(\sum_{B_1} \chi_{B_1} L_{B_1} \right)^\dagger \left(\sum_{B_2} \chi_{B_2} L_{B_2} \right) \right\rangle \\ &\geq 0, \end{aligned} \quad (48)$$

because this quantity is an expectation value of the positive operator. Therefore, the coefficients $C_{B_1 B_2}^\Delta$ form a positive matrix. The master equation (45) is the same one derived in Refs. 5 and 12-14.

3. Summary

In this paper, by the renormalization group method, we have rederived the Markovian master equation in the GKLS form under the following assumptions:

- (1) The interaction between the system and the bath is weak (weak coupling) and the back action of the system on the bath is negligible.
- (2) The correlation time of the bath field is sufficiently shorter than the relaxation time of the system and the existence of the factorized state at the specific initial time t_0 can be assumed.
- (3) The coarse graining time Δ is longer than the time scale of the bath field.

The naive perturbative solution Eq. (27) does not have a form of the master equation. To transform it to the master equation, we introduced the coarse graining time scale Δ , which is larger than the bath time scale t_B . This reduces the double integral with respect to time in the solution to the single integral, and this reduction is equivalent to introducing the Markovian approximation carried out in conventional derivations of the master equation. However, at this stage, the equation does not have the form of the master equation. Then, we applied the renormalization group method and eliminated a specific initial time. Conventional derivations of the master equation usually do the same thing by requiring the time translational

invariance of the considering system and the structure of the equation does not depend on the specific initial time. We have carried out the equivalent procedure by assuming that the structure of the evolution does not depend on the specific renormalization point and derived the renormalization group equation, which provides the desired master equation.

The obtained renormalization group equation generates a dynamical semigroup for the renormalized constant, and the renormalized density operator for the slow time variable τ satisfies the time local Markovian master equation in Lindblad form. As discussed in Refs. 5 and 12-14, the master equation (45) reduces to the one with the rotational wave approximation and the secular approximation in the limit $\Delta \rightarrow \infty$. However, for the purpose of detection of the entanglement of the quantum field using particle detectors, the master equation with the time coarse graining is more suitable because it can detect the vacuum quantum fluctuation, which is the cause of the entanglement of the quantum field.

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